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Notes, Comments, and Letters to the Editor

Sensitivity of Optimal Programs with Respect to Changes in Target Stocks: The Case of Irreversible Investment*

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Consider an aggregative model of economic growth with changing technology and tastes, in which investment is irreversible. It is shown that initial decisions in finite-horizon optimal programs are insensitive to changes in terminal stocks, provided the horizon is long enough. This generalizes Brock's result, which was proved assuming investment to be reversible. The irreversibility constraint does not allow one to follow Brock's method of proof, using the dual (Shadow Price) properties of optimal programs. An alternative method of proof is developed, using a primal approach, and exploiting dynamic programming arguments. *Journal of Economic Literature* Classification Number: 111.

1. INTRODUCTION

An interesting problem in the theory of optimal allocation of resources over a finite-horizon is to examine the sensitivity of such allocations to the level of terminal (end of horizon) targets.

Restricting our attention to an aggregative model of economic growth (with changing technology and tastes), the definitive treatment of this problem is contained in a paper by Brock [1].[1] (Simultaneously, and quite independently, Mirrlees and Hammond [9] also arrived at several of the results contained in Brock's paper).

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[1] For an earlier treatment of this problem, using a linear production function and quasi-stationary utility functions, see Chakravarty [3]. For multisectoral versions of sensitivity results, see Hammond [5], McKenzie [6], and Nermuth [10]. For a stochastic version of Brock's results, see Brock and Mirman [2].

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One important assumption in Brock's framework is that the existing amount of the single good, whether in the form of current output or depreciated capital stock, can be totally consumed. In other words, it is feasible to run down the capital stock at any rate we wish and enjoy a corresponding increase in consumption.

However, it is clearly more realistic to assume that investment should be "irreversible," in the sense that gross investment (net increase in capital plus the amount of depreciation) should be non-negative at each date. A natural question that arises then is the following: "If we incorporate the aspect of irreversibility of investment into Brock's framework, do the sensitivity results he has obtained continue to be valid?"[2]

The answer to the above question turns out to be: "Yes." This answer in itself is of interest, since it demonstrates the robustness of Brock's results. But more intersting, I believe, is the method which leads to this answer. It should be noted that Brock's method of proof, relying as it does on the property that optimal programs satisfy "Euler equations," runs into difficulties with the added "irreversible investment" constraint, since there is now no way to avoid optimal solutions from being "corner solutions," at least for some dates in the finite-horizon.

A recent contribution on the optimal allocation of resources under non-convexities in production by Dechert and Nishimura [4] suggests a technique which enables one to obtain the sensitivity results in the case of irreversible investment. Simultaneously, one can also obtain Brock's results in the case of "reversible investment" as a special case.[3] This technique, using "value functions" associated with finite-horizon optimization problems (together with dynamic programming arguments) rather than the dual (shadow price) properties of such optimization problems, thus appears to be more general and convenient to solve the "sensitivity problem."[4]

2. The Model

2a. *Production*

Consider an aggregative framework with changing technology, given by a sequence of production functions, g_t (where $t = 0, 1, 2, 3, \ldots$) from R_+ to

[2] This question is briefly addressed in Brock ([1, pp. 75–76, footnote 4]). Brock conjectures that his results will continue to be valid, although a different method than his might have to be used.

[3] This technique is also suggested by Brock's remarks in footnotes 4 and 5 on pages 75–78 of his paper [1].

[4] Of course, from the point of view of decentralization of planning, the development of a duality theory of such optimization problems is clearly an important one. This has been done in a paper by Mitra and Ray [8].

itself. Given a nonnegative capital input x in period t, it is possible to produce a "current" output y in period $(t+1)$, where $y = g_t(x)$.

The following assumtions on g_t are used:

(A.1) *For* $t \geqslant 0$, $g_t(0) = 0$.

(A.2) *For* $t \geqslant 0$, g_t *is strictly increasing.*

(A.3) *For* $t \geqslant 0$, $g_t(x)$ *is concave and continuous for* $x \geqslant 0$.

The initial capital input, $\mathbf{x}$, is considered to be historically given and positive. Capital stock is considered to depreciate at a constant rate, d, where $0 \leqslant d \leqslant 1$.[5] Then we can define a sequence of *total ouput functions*, f_t (where $t = 0, 1, 2, \ldots$) from R_+ to itself by

$$f_t(x) = g_t(x) + (1-d)x \qquad \text{for} \quad x \geqslant 0. \tag{2.1}$$

A *feasible program* is a sequence $\langle x \rangle = \langle x_t \rangle$ satisfying

$$\begin{aligned} x_0 = \mathbf{x}, \ 0 \leqslant x_{t+1} \leqslant f_t(x_t) \qquad &\text{for} \quad t \geqslant 0, \\ x_{t+1} \geqslant (1-d)x_t \qquad &\text{for} \quad t \geqslant 0. \end{aligned} \tag{2.2}$$

Associated with a feasible program $\langle x \rangle$ is a *consumption* sequence $\langle c \rangle = \langle c_t \rangle$ given by

$$c_{t+1} = f_t(x_t) - x_{t+1} \qquad \text{for} \quad t \geqslant 0, \tag{2.3}$$

and a gross *investment* sequence $\langle z \rangle = \langle z_t \rangle$ given by

$$z_{t+1} = x_{t+1} - (1-d)x_t \qquad \text{for} \quad t \geqslant 0. \tag{2.4}$$

Combining the information given by (2.1)–(2.4), we note that current output is either consumed or invested:

$$g_t(x_t) = c_{t+1} + z_{t+1} \qquad \text{for} \quad t \geqslant 0 \tag{2.5}$$

and both consumption and gross investment are non-negative at each date.[6]

[5] There is, of course, no difficulty in accomodating the case where the depreciation rate is also variable over time. This does not change any of the results or the methods of proof, as the reader can easily check.

[6] The present framework includes the neoclassical growth model as a special case, by a suitable interpretation of variables. Let $G_t(X, L)$ be a constant return to scale production function, defined on Capital (X) and Labor (L), available at date t. Labor is assumed to grow exogenously at a rate $n \geqslant 0$, i.e., $L_t = L_0(1+n)^t$ for $t \geqslant 0$; $L_0 > 0$. Capital depreciates at a constant rate $\bar{d}$, where $0 \leqslant \bar{d} \leqslant 1$. The basic neoclassical growth equation is, $G_t(X_t, L_t) = C_{t+1} + X_{t+1} - (1-\bar{d})X_t = C_{t+1} + Z_{t+1}$. Dividing through by L_{t+1}, and denoting (X_t/L_t) by x_t, (C_{t+1}/L_{t+1}) by c_{t+1}, (Z_{t+1}/L_{t+1}) by z_{t+1}, for $t \geqslant 0$, we have $[G_t(x_t, 1)/(1+n)] = c_{t+1} + z_{t+1} = c_{t+1} + (x_{t+1} - [(1-\bar{d})/(1+n)]x_t)$. Then letting $g_t(x) = G_t(x, 1)/(1+n)$ and $d = (n+\bar{d})/(1+n)$, we obtain the framework described in Section 2.

The feasible program $\langle \bar{x}_t \rangle$ given by

$$\bar{x}_0 = \mathbf{x}, \ \bar{x}_{t+1} = f_t(\bar{x}_t) \qquad \text{for} \quad t \geqslant 0 \tag{2.6}$$

is called the *pure accumulation program.* Clearly, for every feasible program $\langle x_t \rangle$, we have

$$(x_t, c_t, z_t) \leqslant (\bar{x}_t, \bar{x}_t, \bar{x}_t) \qquad \text{for} \quad t \geqslant 1. \tag{2.7}$$

A feasible program $\langle x_t \rangle$ is *inefficient* if there is a feasible program $\langle x_t' \rangle$ such that $c_t' \geqslant c_t$ for all t, and $c_t' > c_t$ for some t. It is called *efficient* if it is not inefficient.

For a positive integer, T, and a non-negative real number, b,[7] a *T-program to b* is a finite sequence $\langle x_t(T, b) \rangle$ satisfying

$$\begin{aligned} &x_0(T, b) = \mathbf{x}; \ 0 \leqslant x_{t+1}(T, b) \leqslant f_t(x_t(T, b)) \qquad \text{for} \quad 0 \leqslant t \leqslant T-1, \\ &x_{t+1}(T, b) \geqslant (1-d)\, x_t(T, b) \qquad \text{for} \quad 0 \leqslant t \leqslant T-1; \ x_T(T, b) \geqslant b. \end{aligned} \tag{2.8}$$

Associated wth a T–program to b is a finite *consumption* sequence $\langle c_t(T, b) \rangle$ given by

$$c_{t+1}(T, b) = f_t(x_t(T, b)) - x_{t+1}(T, b) \qquad \text{for} \quad 0 \leqslant t \leqslant T-1, \tag{2.9}$$

and a finite gross *investment* sequence $\langle z_t(T, b) \rangle$ given by

$$z_{t+1}(T, b) = x_{t+1}(T, b) - (1-d)\, x_t(T, b) \qquad \text{for} \quad 0 \leqslant t \leqslant T-1. \tag{2.10}$$

2b. *Preferences*

The preferences of the planner will be represented by a sequence of utility functions,[8] u_t (where $t = 1, 2, \ldots$) from R_+ to R.[9] The following assumptions on u_t will be used:

(A4) *For $t \geqslant 1$, u_t is strictly increasing.*

(A.5) *For $t \geqslant 1$, u_t is continuous and strictly concave for $c \geqslant 0$.*

[7] Here, T is to be interpreted as the (finite) planning horizon and b the target (end of horizon) capital stock.

[8] The important case treated in optimal growth theory is one in which there is a utility function, u, from R_+ to R, and a *discount factor* $\sigma > 0$, such that $u_t(c) = \sigma^{t-1} u(c)$ for $t \geqslant 1$. This is clearly covered as a special case of our analysis.

[9] This means that $u_t(0)$ is finite. Brock [1] assumes that $u_t(0) = -\infty$. This case can be incorporated in our analysis by allowing the range of the utility functions to be the extended real line. Apart from some minor changes in the statements of the results, and in the definition of the value function, the analysis remains unaffected.

A T-program to b, $\langle x_t^*(T,b)\rangle$, is an *optimal T-program* to b if

$$\sum_{t=1}^{T} u_t(c_t^*(T,b)) \geqslant \sum_{t=1}^{T} u_t(c_t(T,b)) \tag{2.11}$$

for every T-program to b, $\langle x_t(T,b)\rangle$.

A feasible program $\langle x_t^*\rangle$ is a *weakly maximal* program if

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [u_t(c_t) - u_t(c_t^*)] \leqslant 0 \tag{2.12}$$

for every feasible program $\langle x_t\rangle$.

3. Some Preliminary Results

In this section we have collected some results which will permit the "sensitivity analysis" of the next section easier to present. These results are fairly straightforward to prove, but the proofs, written out fully, turn out to be long and tedious. Hence, all proofs in this section have been omitted.[10]

The first result establishes a boundedness property for T-programs.

Lemma 3.1. *If $\langle x_t(T,b)\rangle$ is a T-program to b, then*

$$(x_t(T,b), c_t(T,b), z_t(T,b)) \leqslant (\bar{x}_t, \bar{x}_t, \bar{x}_t), \qquad 1 \leqslant t \leqslant T. \tag{3.1}$$

The second result establishes the existence and uniqueness of an optimal T-program to b.

Lemma 3.2. *There exists an optimal T-program to b if $b \leqslant \bar{x}_T$. Furthermore, an optimal T-program to b is unique.*

At this point, it is useful to introduce the concept of a value function. We define the *value function*, $V(T,b)$ by

$$V(T,b) = \sup\left[\sum_{t=1}^{T} u_t(c_t(T,b)) \colon \langle x_t(T,b)\rangle \text{ is a } T\text{-program to } b, \text{ where } T \geqslant 1, \text{ and } 0 \leqslant b \leqslant \bar{x}_T\right].$$ [11]

[10] The proofs can be obtained from the author, on request.

[11] It is understood that the supremum is taken over all T-programs to b.

By Lemma 3.2, there is a T-program to b, $\langle x_t^*(T,b)\rangle$, such that $V(T,b)=\sum_{t=1}^{T} u_t(c_t^*(T,b))$.

The next three results state the different implications of what is known as the "principle of optimality."

LEMMA 3.3. *If* $\langle x_t^*(T,b)\rangle$ *is the optimal T-program to b, then for* $1\leqslant s\leqslant T$, $(x_0^*(T,b),\ldots,x_s^*(T,b))$ *is the optimal s-program to* $x_s^*(T,b)$.

LEMMA 3.4. *If* $\langle x_t^*(T,b)\rangle$ *is the optimal T-program to b, and* $T\geqslant 2$, *then*

$$V(T,b)=V(T-1,x_{T-1}^*(T,b))+u_T(c_T^*(T,b)). \tag{3.2}$$

LEMMA 3.5. *If* $\langle x_t(T,b)\rangle$ *is a T-program to b, and* $T\geqslant 2$, *then*

$$V(T,b)\geqslant V(T-1,x_{T-1}(T,b))+u_T(c_T(T,b)). \tag{3.3}$$

4. SENSITIVITY ANALYSIS

In this section, we follow Brock [1] in his four-step procedure to arrive at the sensitivity result. These four steps may be summarized as follows[12]:

(1) If we compare the T-optimal programs to two different target stocks, we find that at each date the input level of the T-optimal program with the larger target stock is larger, compared to the input level of the T-optimal program with the smaller target stock (Theorem 4.1).

(2) If we compare the T-optimal program and the $T+1$-optimal program to the zero target stock, then at each date (up to T) the input level of the $T+1$-optimal program is larger than the input level of the T-optimal program (Theorem 4.2).

(3) There is a feasible program (called the "limit program" henceforth) such that at each date, the input level of the T-optimal program to the zero target stock increases monotonically, and converges to the input level of the limit program as the horizon (T) increases to infinity (Theorem 4.3).

(4) At each date, the input level of the T-optimal program to the target stock of b converges to the input level of the limit program, as the horizon (T) increases to infinity, provided the target stock of b is smaller

[12] In summarizing the four-step procedure, we have used words like "larger" and "smaller" loosely; that is, without distinguishing between weak and strict inequalities. Precise statements are contained in the Theorems.

than the "smallest limit point" of the input levels of the limit program (Theorem 4.4).

Results (3) and (4) together imply the following sensitivity result: the first period input level choice on a T-optimal program to a target stock b, changes "little" with a change in the target stock b (provided b satisfies the bound specified in (4)) if the horizon T is "large enough."[13]

THEOREM 4.1[14]. *If $\langle x_t(T, b)\rangle$ is the optimal T-program to b, and $\langle x_t(T, b')\rangle$ is the optimal T-program to b', and $b \geqslant b'$, then*

$$x_t(T, b) \geqslant x_t(T, b') \qquad \text{for} \quad 0 \leqslant t \leqslant T. \tag{4.1}$$

Proof. We first show that

$$x_T(T, b) \geqslant x_T(T, b'). \tag{4.2}$$

It should be mentioned that this is not obvious in the case of irreversible investment, since $x_T(T, b)$ need not equal b, and $x_T(T, b')$ need not equal b'. Suppose (4.2) is violated; then, $x_T(T, b') > x_T(T, b) \geqslant b \geqslant b'$. Then, clearly, $\langle x_t(T, b)\rangle$ is a T-program to b', so $V(T, b') \geqslant V(T, b)$. Also, $\langle x_t(T, b')\rangle$ is a T-program to b, so $V(T, b') \leqslant V(T, b)$. Hence, we must have $V(T, b) = V(T, b')$, which proves that $\langle x_t(T, b)\rangle$ is an optimal T-program to b'. This contradicts Lemma 3.2, since $\langle x_t(T, b')\rangle$ is an optimal T-program to b', and $x_T(T, b') \neq x_T(T, b)$. This establishes (4.2).

To prove (4.1), suppose this is violated for some t. Let s be the last period for which the violation occurs. Then, by (4.2), $s \leqslant T - 1$, so $s + 1 \leqslant T$, and $x_{s+1}(T, b) \geqslant x_{s+1}(T, b')$, while $x_s(T, b) < x_s(T, b')$. By Lemma 3.3, $(x_0(T, b),..., x_{s+1}(T, b))$ is the optimal $(s + 1)$-program to $x_{s+1}(T, b)$; similarly, we know that $(x_0(T, b'),..., x_{s+1}(T, b'))$ is the optimal $(s + 1)$-program to $x_{s+1}(T, b')$.

If $x_{s+1}(T, b) = x_{s+1}(T, b')$, then by Lemma 3.2, $x_s(T, b) = x_s(T, b')$, a contradiction. Thus, we must have $x_{s+1}(T, b) > x_{s+1}(T, b')$. Now, using Lemma 3.4, we have

$$V(s + 1, x_{s+1}(T, b)) = V(s, x_s(T, b)) + u_{s+1}(c_{s+1}(T, b)). \tag{4.3}$$

Consider the finite sequence $(x'_0,..., x'_{s+1})$ given by: $x'_t = x_t(T, b')$ for $t = 0,..., s$; $x'_{s+1} = x_{s+1}(T, b)$. Then, $x'_{s+1} = x_{s+1}(T, b) > x_{s+1}(T, b') \geqslant (1 - d)\, x_s(T, b') = (1 - d)x'_s$. Also $x'_{s+1} = x_{s+1}(T, b) \leqslant f_s(x_s(T, b)) <$

[13] Of course, the result does not apply only to the *first* period input level choice, but to any fixed finite number of periods $(1, 2,..., k)$.

[14] The proof of Theorem 4.1 leans heavily on the technique used in Dechert and Nishimura [4, Theorem 1].

$f_s(x_s(T, b')) = f_s(x_s')$. Hence, $(x_0', \ldots, x_{s+1}')$ is a $(s+1)$-program to $x_{s+1}(T, b)$. So, by Lemma 3.5,

$$V(s+1, x_{s+1}(T, b)) \geqslant V(s, x_s(T, b')) + u_{s+1}[f_s(x_s(T, b')) - x_{s+1}(T, b)]. \quad (4.4)$$

Using Lemma 3.4, we also have

$$V(s+1, x_{s+1}(T, b')) = V(s, x_s(T, b')) + u_{s+1}(c_{s+1}(T, b')). \quad (4.5)$$

Consider the finite sequence $(x_0'', \ldots, x_{s+1}'')$ given by: $x_t'' = x_t(T, b)$ for $t = 0, \ldots, s$; $x_{s+1}'' = x_{s+1}(T, b')$. Then, $x_{s+1}'' = x_{s+1}(T, b') \geqslant (1-d)\, x_s(T, b') \geqslant (1-d)\, x_s(T, b) = (1-d)x_s''$. Also, $x_{s+1}'' = x_{s+1}(T, b') < x_{s+1}(T, b) \leqslant f_s(x_s(T, b)) = f_s(x_s'')$. Hence, $(x_0'', \ldots, x_{s+1}'')$ is a $(s+1)$-program to $x_{s+1}(T, b')$. So, by Lemma 3.5,

$$V(s+1, x_{s+1}(T, b')) \geqslant V(s, x_s(T, b)) + u_{s+1}[f_s(x_s(T, b)) - x_{s+1}(T, b')]. \quad (4.6)$$

For convenience, denote $c_{s+1}(T, b)$ by a, $c_{s+1}(T, b')$ by a'; also denote $[f_s(x_s(T, b')) - x_{s+1}(T, b)]$ by e, and $[f_s(x_s(T, b)) - x_{s+1}(T, b')]$ by e'. Then, using (4.3) and (4.5), we have

$$\begin{aligned} &V(s+1, x_{s+1}(T, b)) - V(s, x_s(T, b)) + V(s+1, x_{s+1}(T, b')) - V(s, x_s(T, b')) \\ &\quad = u_{s+1}(a) + u_{s+1}(a'). \end{aligned} \quad (4.7)$$

Also, using (4.4) and (4.6), we have

$$\begin{aligned} &V(s+1, x_{s+1}(T, b)) - V(s, x_s(T, b')) + V(s+1, x_{s+1}(T, b')) - V(s, x_s(T, b)) \\ &\quad \geqslant u_{s+1}(e) + u_{s+1}(e'). \end{aligned} \quad (4.8)$$

Combining (4.7) and (4.8), we have

$$u_{s+1}(a) + u_{s+1}(a') \geqslant u_{s+1}(e) + u_{s+1}(e'). \quad (4.9)$$

By definition of e and e', we have

$$\begin{aligned} e + e' &= f_s(x_s(T, b')) - x_{s+1}(T, b') + f_s(x_s(T, b)) - x_{s+1}(T, b) \\ &= c_{s+1}(T, b') + c_{s+1}(T, b) = a + a'. \end{aligned} \quad (4.10)$$

Since $x_s(T, b') > x_s(T, b)$, and $x_{s+1}(T, b) > x_{s+1}(T, b')$, so $a < e < a'$. Hence, there is a real number, θ, such that $0 < \theta < 1$, and $e = \theta a + (1-\theta)a'$. Then, using (4.10), we also have $e' = (1-\theta)a + \theta a'$. Thus, we have $u_{s+1}(e) >$

$\theta u_{s+1}(a) + (1-\theta)\, u_{s+1}(a')$; also, $u_{s+1}(e') > (1-\theta)\, u_{s+1}(a) + \theta u_{s+1}(a')$. Combining these last two inequalities, we have

$$u_{s+1}(e) + u_{s+1}(e') > u_{s+1}(a) + u_{s+1}(a'). \tag{4.11}$$

But (4.11) contradicts (4.9), and this establishes the theorem.

THEOREM 4.2. *If $\langle x_t(T, 0)\rangle$ is the T-optimal program to zero, and $\langle x_t(T+1, 0)\rangle$ is the $(T+1)$-optimal program to zero, then*

$$x_t(T, 0) \leqslant x_t(T+1, 0) \qquad \textit{for} \quad 0 \leqslant t \leqslant T \tag{4.12}$$

Proof. We will show that

$$x_T(T, 0) \leqslant x_T(T+1, 0). \tag{4.13}$$

Suppose that (4.13) is violated. Consider the finite sequence $(x'_0,\dots, x'_{T+1})$ given by: $x'_t = x_t(T, 0)$ for $t = 0,\dots, T$; $x'_{T+1} = (1-d)x'_T$. Then, $x'_{T+1} = (1-d)\, x_T(T, 0) < f_T(x_T(T, 0)) = f_T(x'_T)$. Thus, $(x'_0,\dots, x'_{T+1})$ is a $(T+1)$-program to zero, and $c'_t = c_t(T, 0)$ for $t = 0,\dots, T$; $c'_{T+1} = f_T(x_T(T, 0)) - (1-d)\, x_T(T, 0) = g_T(x_T(T, 0)) > g_T(x_T(T+1, 0)) \geqslant c_{T+1}(T+1, 0)$.

Now, $(x_0(T+1, 0),\dots, x_T(T+1, 0))$ is a T-program to zero, so

$$\sum_{t=1}^{T} u_t(c_t(T, 0)) \geqslant \sum_{t=1}^{T} u_t(c_t(T+1, 0)). \tag{4.14}$$

Using (4.14), we then have

$$\sum_{t=1}^{T+1} u_t(c'_t) > \sum_{t=1}^{T+1} u_t(c_t(T+1, 0)). \tag{4.15}$$

But (4.15) contradicts the fact that $\langle x_t(T+1, 0)\rangle$ is the $(T+1)$-optimal program to zero, and this establishes (4.13). Now, (4.12) follows directly, by using Theorem 4.1.

THEOREM 4.3. *There exists a unique feasible program $\langle x_t^*\rangle$, such that if $\langle x_t(T, 0)\rangle$ is the T-optimal program to zero, then for $t \geqslant 0$,*

$$x_t(T, 0) \to x_t^* \qquad \textit{as} \quad T \to \infty. \tag{4.16}$$

Proof. For each $t \geqslant 0$, we have $x_t(T, 0) \leqslant x_t(T+1, 0)$ by Theorem 4.2. Hence, $x_t(T, 0)$ is monotonically non-decreasing in T. By Lemma 3.1, $x_t(T, 0) \leqslant \bar{x}_t$ for all T. Hence $x_t(T, 0)$ converges to some non-negative real number, call it x_t^*, as T increases to infinity. Using (2.2) and (2.8) it is straightforward to check that $\langle x_t^* \rangle$ is a feasible program. The uniqueness of $\langle x_t^* \rangle$ is clear from the above proof.

Given the feasible program $\langle x_t^* \rangle$, we define $b^* = \liminf_{t \to \infty} x_t^*$.

THEOREM 4.4[15]. *If* $0 \leqslant b < b^*$, *and* $\langle x_t(T, b) \rangle$ *is the T-optimal program to b, then for each* $t \geqslant 0$,

$$x_t(T, b) \to x_t^* \quad \text{as} \quad T \to \infty. \tag{4.17}$$

Proof. By definition of b^*, there is T_0, such that $x_t^* > b$ for $t \geqslant T_0$. By Theorem 4.3, for each $T \geqslant T_0$, there is a positive integer N (depending on T), such that $N \geqslant T$, and $x_T(N, 0) > b$. By Lemma 3.3, $(x_0(N, 0), \ldots, x_T(N, 0))$ is the T-optimal program to $x_T(N, 0)$. Also, $\langle x_t(T, b) \rangle$ is the T-optimal program to b. So, by Theorem 4.1, we have $x_t(T, b) \leqslant x_T(N, 0)$ for $0 \leqslant t \leqslant T$. Also by Theorem 4.1, $x_t(T, b) \geqslant x_t(T, 0)$ for $0 \leqslant t \leqslant T$. Combining these last two pieces of information, we have

$$x_t(T, 0) \leqslant x_t(T, b) \leqslant x_t(N, 0) \quad \text{for} \quad 0 \leqslant t \leqslant T. \tag{4.18}$$

Let $T \to \infty$; then $N \to \infty$ also. By Theorem 4.3, for each t, $x_t(T, 0) \to x_t^*$, and $x_t(N, 0) \to x_t^*$. So, by (4.18), $x_t(T, b) \to x_t^*$ as $T \to \infty$. This establishes (4.17).

5. WEAK-MAXIMALITY OF THE LIMIT PROGRAM

It is of interest to know whether the limit program established in Section 4 is weakly maximal. More generally, it would be useful to know the relationship between the limit program and a weakly maximal program (provided one exists).

It is straightforward to provide an example in which the limit program is, in fact, inefficient and, hence, not weakly maximal.

EXAMPLE 5.1. Let $g_t(x) = x$, for $t \geqslant 0$, $u_t(c) = c/(1 + c)$ for $t \geqslant 1$, $d = 1$, $\mathbf{x} = 1$. Given any $T \geqslant 1$, the T-optimal program to zero $\langle x_t(T, 0) \rangle$ must satisfy: $c_t(T, 0) = 1/T$ for $t = 1, \ldots, T$. Thus the limit program $\langle x_t^* \rangle$ must satisfy $c_t^* = 0$ for $t \geqslant 1$. Thus, the limit program is inefficient and hence not weakly maximal.

The wary reader might not be convinced by Example 5.1, since $d = 1$

[15] By putting $d = 1$ in the present framework, we obtain the mathematical form of the model studied by Brock [1]. (This statement should not, of course, be interpreted to mean that in Brock's model, capital is assumed to be completely non-durable). Hence, the sensitivity result obtained in this note is a generalization of Brock's result, and his sensitivity result can be obtained as a corollary of Theorem 4.4.

implies that we are really in the reversible world. The next example settles the issue, for the "genuine" irreversible model.

EXAMPLE 5.2. Let $g_t(x) = x/2$ for $t \geq 0$, $u_t(c) = 20^t[c/(1+c)]$, $d = \frac{1}{2}$, $\mathbf{x} = 1$. Given $T \geq 2$, a T-optimal program to zero $\langle x_t(T, 0)\rangle$ must satisfy: $c_t(T, 0) = 0$ for $t = 1, ..., T-1$, and $c_T(T, 0) = \frac{1}{2}$. Thus, the limit program $\langle x_t^*\rangle$ must satisfy: $c_t = 0$ for $t \geq 1$. Thus, the limit program is inefficient and hence not weakly maximal.

Brock [1] showed that when investment is reversible, if there does exist a weakly maximal program then it must be the limit program. (See [1, Corollary 2, p. 80].) This result remains valid in our framework (see Theorem 5.1 below), but Brock's method of proof has to be abandoned. In order to establish his Corollary 2, Brock proved the following result: if $\langle x_t(T, b)\rangle$ is a T-optimal program to b and $\langle x_t(T, b')\rangle$ is a T-optimal program to b', and $b \geq b'$, then $c_t(T, b) \leq c_t(T, b')$, $t = 1, ..., T$. (See [1, Theorem 1, p. 77].) When investment is irreversible, this monotone property of *consumption* no longer holds (though that of *inputs* holds, as we verified in Theorem 4.1). An example settling this issue appears in Majumdar and Nermuth [7].[16] Thus, we are forced to use a new method of proof, which, fortunately, turns out to be fairly simple.

THEOREM 5.1. *If there exists a weakly maximal program* $\langle \tilde{x}_t\rangle$, *then* $\tilde{x}_t = x_t^*$ *for* $t \geq 1$, *where* $\langle x_t^*\rangle$ *is the limit program obtained in Theorem* 4.3.

Proof. Denote, for $T \geq 1$, the T-optimal program to zero by $\langle \tilde{x}_t(T, 0)\rangle$.

First, we claim that for each $T \geq 1$, $\tilde{x}_T \geq \tilde{x}_T(S, 0)$ for $S \geq T$. Pick any $T \geq 1$. Then for $S \geq T$, $(\tilde{x}_0, ..., \tilde{x}_S)$ is the S-optimal program to $\tilde{x}_S$. Since $\tilde{x}_S \geq 0$, so by Theorem 4.1, $\tilde{x}_t \geq \tilde{x}_t(S, 0)$ for $t = 1, ..., S$. In particular, $\tilde{x}_T \geq \tilde{x}_T(S, 0)$.

Next, we claim that for each $T \geq 1$,

$$\sum_{t=1}^{T} u_t(c_t^*) \geq \sum_{t=1}^{T} u_t(\tilde{c}_t). \tag{5.1}$$

Pick any $T \geq 1$. Then for $S \geq T$, $(\tilde{x}_0(S, 0), ..., \tilde{x}_T(S, 0))$ is a T-optimal program to $\tilde{x}_T(S, 0)$. Also $\tilde{x}_T \geq \tilde{x}_T(S, 0)$ by the above argument, so

$$\sum_{t=1}^{T} u_t(\tilde{c}_t(S, 0)) \geq \sum_{t=1}^{T} u_t(\tilde{c}_t). \tag{5.2}$$

[16] This paper also contains an excellent analysis of sensitivity and turnpike (asymptotic stability) results when investment is irreversible and there are non-convexities in the production set.

Since $\tilde{c}_t(S, 0) \to c_t^*$ (for $t = 1,..., T$) as $S \to \infty$, so by (5.2), (5.1) follows immediately.

Next, we claim that $\langle x_t^* \rangle$ is weakly maximal. If not, there is a feasible program $\langle x_t \rangle$, a positive number α, and an integer $T^* \geqslant 1$, such that

$$\sum_{t=1}^{T} u_t(c_t) \geqslant \sum_{t=1}^{T} u_t(c_t^*) + \alpha \qquad \text{for} \quad T \geqslant T^*. \tag{5.3}$$

Using (5.1) and (5.3) contradicts the weak-maximality of $\langle \tilde{x}_t \rangle$. Thus $\langle x_t^* \rangle$ is weakly maximal.

Now, suppose $c_t^* \neq \tilde{c}_t$ for some $t = s$. Then, consider a sequence $x_t = \frac{1}{2}(x_t^* + \tilde{x}_t)$, for $t \geqslant 0$. Then $\langle x_t \rangle$ is a feasible program, and $c_t \geqslant \frac{1}{2}(c_t^* + \tilde{c}_t)$ for $t \geqslant 1$. Consequently, $u(c_t) - u(\tilde{c}_t) \geqslant \frac{1}{2}u(c_t^*) + \frac{1}{2}u(\tilde{c}_t) - u(\tilde{c}_t) = \frac{1}{2}[u(c_t^*) - u(\tilde{c}_t)]$ for $t \geqslant 1$; also, $u(c_t) - u(\tilde{c}_t) > \frac{1}{2}u(c_t^*) + \frac{1}{2}u(\tilde{c}_t) - u(c_t) = \frac{1}{2}[u(c_t^*) - u(\tilde{c}_t)]$ for $t = s$. Thus there is $T^* \geqslant 1$, and $\alpha > 0$, such that

$$\sum_{t=1}^{T} [u(c_t) - u(\tilde{c}_t)] \geqslant \alpha + \frac{1}{2} \sum_{t=1}^{T} [u(c_t^*) - u(\tilde{c}_t)] \qquad \text{for} \quad T \geqslant T^*.$$

Using (5.1) contradicts the weak maximality of $\langle \tilde{x}_t \rangle$. Thus $c_t^* = \tilde{c}_t$ for $t \geqslant 1$, and so $x_t^* = \tilde{x}_t$ for $t \geqslant 0$.

As a final remark, we note that Brock [1] showed that when investment is reversible, if the limit program is efficient, it is weakly maximal (see his [1, Corollary 3, p. 81], and his remarks following it on p. 82). This result exploited the shadow price properties of optimal and efficient programs. In our framework, we feel that this issue cannot be explored fully until a complete duality theory is developed for optimal and efficient programs when investment is irreversible. This, of course, is beyond the scope of this note. We refer the reader to the paper by Mitra and Ray [8], which addresses this and related issues.

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